Review of complex numbers

Two central ideas in cryo-EM image analysis are best expressed using complex numbers. First, the quantum-mechanical description of an electron wave is a complex exponential function, and we will use this representation to understand phase-contrast imaging in the electron microscope. Second, the Fourier transform (FT) is central to 3D reconstruction algorithms, and the simplest description of the FT uses complex numbers. The point of this handout is to review the basics of complex numbers so we can use them in succeeding lectures.

Imaginary number
The unit imaginary number is denoted i. It is defined as

$$i = \sqrt{-1}$$

and some powers of *i* are

$$\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \end{aligned}$$

It has an absolute value (as defined below) of

$$abs(i) = 1$$

Complex numbers
A complex number is the sum of real and an imaginary part. So for example two complex numbers *z* and *w* might be

$$\begin{aligned} z &= a + bi \\ w &= c + di \end{aligned}$$

Where the coefficients *a,b,c* and *d* are ordinary real numbers. The product of these complex numbers is gotten by

$$\begin{aligned} zw &= (a + bi)(c + di) \\ &= ac - bd + (ad + bc)i \end{aligned}$$

where you'll notice that there is a minus sign in front of the *bd* term because $i^2 = -1$.

The functions Re and Im extract the real and imaginary parts of a complex number. Thus for *z* as defined above,

$$\begin{aligned} \mathrm{Re}(z) &= a \\ \mathrm{Im}(z) &= b. \end{aligned}$$

The absolute value of a complex number is the root-sum-of-squares of the real and imaginary parts,

$$\mathrm{abs}(z) = \sqrt{a^2 + b^2}$$

Complex exponential

A very important function in physics is the exponential function e^x where e is the special number 2.71828… and is called the base of the natural logarithms. An infinite series expansion for e^x is given by

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

and later on we'll make a lot of use of the approximation $e^x \approx 1 + x$.

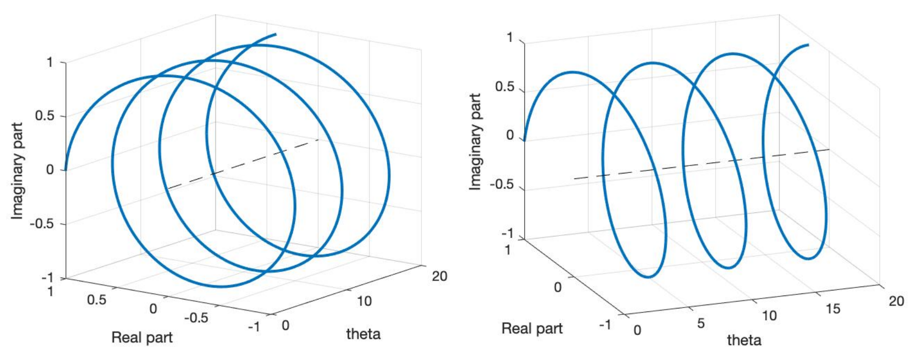
Recall that exponential functions obey a product rule

$$e^{x+y} = e^x e^y$$

All of these properties hold just as well when the exponential function has a complex argument. The most famous example is

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This is a function that is periodic with period 2π and if you plot it in three dimensions it traces out a helix. Below are two views of such a plot.



It always has unity absolute value, $\text{abs}\left(e^{i\theta}\right) = 1$

Beside expressing a complex number by its real and imaginary parts, it's possible to instead express it by its magnitude and phase, A and θ. For our variable z we can write

$$z = Ae^{i\theta}$$

where $A = a\text{bs}(z)$ and $\theta = \tan^{-1}(b/a)$

Complex conjugate

A special operation you can do to a complex number is to flip the sign of the imaginary part. The result is called the complex conjugate, and it's denoted by a star. So if $z = a + bi$, then the conjugate $z^* = a - bi$. Maybe it doesn't look like a big deal, but the complex conjugate comes up in Fourier transforms in interesting contexts such as reflections and time reversals.

Two little exercises

1. What are the real and imaginary parts of $e^{i\theta} + e^{-i\theta}$?
2. Derive the trigonometric identity $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$ (hint: use the product rule, $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ and take the real part).